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Riemannian Superspaces, Exact Solutions and the Geometrical Meaning of the Field Localization

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Abstract The geometrical origin of a special type of non-degenerate supermetric is elucidated and the connection with processes of topological origin in high energy physics is explained. The new mechanism of the localization of the fields in a particular sector of the supermanifold is explained and the similarity and differences with a 5-dimensional warped model are shown. The relation with gauge theories of supergravity based in the super $SL(2,C)$ group is explicitly given and the possible original action is presented. From the point of view of the vacuum solutions, the simplest Riemannian superspaces are described.

Keywords Supermanifolds · Coherent states · Localization · Confinement

1 Introduction and Motivation

Several attempts have been made by various groups to construct the theory of supergravity as the geometry of a superspace possessing non-zero curvature and torsion tensors without undesirable higher spin states [1–5]. Only few years after those works, the consistent construction of the superfield supergravity was formulated in the pioneering papers independently by V.I. Ogievetsky and E. Sokatchev [6] and S.J. Gates and W. Siegel [7]. From these times in several areas of the theoretical physics the description of different systems was given in the context of the geometry of supermanifolds and superfields [8, 9]: supergravity and d-branes models with warped supersymmetry [10, 11], super-Landau systems [12], superbrane actions from nonlinearly realized supersymmetries [13], etc.

It is therefore of interest to study the geometry not only of the simplest superspaces, but also the more unusual or non-standard ones and elucidate all the gauge degrees of freedom that they possesses. This fact will clarify and expand the possibilities to construct more

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realistic physical models and new mathematically consistent theories of supergravity. On the other hand, the appearance of supergroups must draw attention to the study of the geometries of the homogeneous superspaces whose groups of motions they are. Another motivation of the study of these Riemannian superspaces is in order to establish some degree of uniqueness in the obtained supersymmetric solutions.

Motivated by the above, we complete our previous work [15] studying and analyzing from the point of view of the possible vacuum solutions, the simplest non trivial supermetric given by Volkov and Pashnev in [14] that was the "starting point" toy model of [15]

$$ds^2 = \omega^\mu \omega_\mu + \mathbf{a}\omega^\alpha \omega_\alpha - \mathbf{a}^* \omega^{\dot\alpha} \omega_{\dot\alpha} \tag{1}$$

This particular non-degenerate supermetric contains the complex parameters $\mathbf{a}$ and $\mathbf{a}^*$ that make it different of other more standard supermetrics. Then, our main task is to find the meaning and the role played by these complex parameters from the geometrical and physical point of view. To this end, we compare the solution of Ref. [15] that was computed in the $N = 1$ four dimensional superspace proposed in [14, 16–19], compactified to one dimension and restricted to the pure time-dependent case with:

(i) the well known solution described in references [20, 21] that was formulated in a superspace $(1 \mid 2)$.
(ii) a multidimensional warped model described in [22], in this case with dependence of the solution on the four dimensional bosonic coordinates.

Our goal is to show that, from the point of view of the obtained solutions, the complex parameters $\mathbf{a}$ *localize* the fields in a specific region of the bosonic part of this special superspace, they explicitly *breakdown* the chiral symmetry when some conditions are required and all these very important properties *remain* although the supersymmetry of the model was completely broken. Also, besides all these highlights, we also show that the obtained vacuum states from the extended supermetric are very well defined in any Hilbert space.

About the geometrical origin of this particularly special metric, we demonstrate that it can be *naturally derived* from a theory of supergravity based in a super $SL(2C)$-valuated connection A. When the symmetry of the super $SL(2C)$ model presented here is explicitly written as a function of its reductive components, a part as (1) appear plus a first order (Dirac-like) fermionic term.

The plan of the paper is as follows: in Sect. 2 we give a brief review, based in a previous work of the author [15], about the $N = 1$ non-degenerate four dimensional superspace proposed by Volkov and Pashnev and its solution. Section 3 is devoted to analyze the relation of the supermetric under consideration with the superspace $(1 \mid 2)$ given explicitly under which conditions one is reduced to the other one from the point of view of the obtained vacuum solutions. The geometrical derivation of the supermetric from a gauge theory of supergravity based in the super $SL(2C)$ group, a *new* superparticle model and the link between the complex parameters $\mathbf{a}$ and $\mathbf{a}^*$ and the cosmological constant Λ are given in Sect. 4. In Sect. 5 a surprising connection between the extended supermetric and multidimensional warped gravity model solutions is shown and some hints of a possible *new mechanism* of the field localization and the idea of confinement is proposed. Finally in Sect. 6 the main results and concluding remarks are given.

2 The Particular Four Dimensional $N = 1$ Superspace

The superspace $(1,3 \mid 1)$ has four bosonic coordinates x^μ and one majorana bispinor: $(t, x^i, \theta^\alpha, \overline{\theta}^{\dot{\alpha}})$. Two possible realizations for this superspace are $\rightarrow$

$$\begin{cases} osp(2,2) \rightarrow Bosonic - Fermionic \\ osp(1/2,\mathbb{R}) \rightarrow Bosonic \end{cases}$$

with the following group structure for the bosonic-fermionic realization

$$\begin{pmatrix} SU(1,1) & Q \\ Q & SU(1,1) \end{pmatrix}$$

We will concentrate our analysis to the superspace $(1,3 \mid 1)$ with extended line element as in [14, 16–19]

$$ds^2 = \omega^\mu \omega_\mu + \mathbf{a}\omega^\alpha \omega_\alpha - \mathbf{a}^* \omega^{\dot{\alpha}} \omega_{\dot{\alpha}} \tag{1}$$

invariant to the following supersymmetric transformations

$$x'_\mu = x_\mu + i\left(\theta^\alpha (\sigma)_{\alpha\dot{\beta}} \overline{\xi}^{\dot{\beta}} - \xi^\alpha (\sigma)_{\alpha\dot{\beta}} \overline{\theta}^{dot\beta}\right), \quad \theta'^\alpha = \theta^\alpha + \xi^\alpha, \quad \overline{\theta'}^{\dot{\alpha}} = \overline{\theta}^{\dot{\alpha}} + \overline{\xi}^{\dot{\alpha}}$$

where the Cartan forms of the group of the supersymmetry are

$$\omega_\mu = dx_\mu - i\left(d\theta\sigma_\mu\overline{\theta} - \theta\sigma_\mu d\overline{\theta}\right), \quad \omega^\alpha = d\theta^\alpha, \quad \omega^{\dot{\alpha}} = \overline{d\theta}^{\dot{\alpha}}$$

The spinorial indices are related as follows (the dotted indices are similarly related, as usual):

$$\theta^\alpha = \varepsilon^{\alpha\beta}\theta_\beta, \quad \theta_\alpha = \theta^\beta \varepsilon_{\beta\alpha}, \quad \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \quad \varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}, \quad \varepsilon_{12} = \varepsilon^{12} = 1$$

The complex constants $\mathbf{a}$ and $\mathbf{a}^*$ in the extended line element are arbitrary. This arbitrarily for the choice of $\mathbf{a}$ and $\mathbf{a}^*$are constrained by the invariance and reality of the interval (1). The solution for the metric in the time dependent case with 3 spatial dimensions compactified (i.e. $\mathbf{R}^1 \otimes S^3$, Ref. [23]) takes the form [15]

$$g_{ab}(t) = e^{A(t)+\xi\varrho(t)} g_{ab}(0) \tag{2}$$

with the following superfield solution

$$\varrho(t) = \phi_\alpha + \overline{\chi}_{\dot{\alpha}}$$

(i.e. chiral plus anti-chiral parts). The system of equations for $A(t)$ and $\varrho(t)$ that we are looking for was given in [15], and is the following

$$\begin{aligned} &|\mathbf{a}|^2 \ddot{A} + m^2 = 0, \\ &\ddot{\overline{\chi}}_{\dot{\alpha}} - i\frac{\omega}{2}(\sigma^0)^\alpha_{\dot{\alpha}}\phi_\alpha = 0, \\ &-\ddot{\phi}_\alpha + i\frac{\omega}{2}(\sigma^0)^{\dot{\beta}}_\alpha \overline{\chi}_{\dot{\beta}} = 0 \end{aligned} \tag{3}$$

The above system can be solved exactly given us the following result

$$A = -\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1 t + c_2, \quad c_1, c_2 \in \mathbb{C} \tag{4}$$

and

$$\phi_\alpha = \overset{\circ}{\phi}_\alpha \left(\alpha e^{i\omega t/2} + \beta e^{-i\omega t/2}\right) + \frac{2i}{\omega}(\sigma^0)_\alpha^{\dot{\beta}} \overline{Z}_{\dot{\beta}}, \tag{5}$$

$$\overline{\chi}_{\dot{\alpha}} = \left(\sigma^0\right)_{\dot{\alpha}}^{\alpha} \overset{\circ}{\phi}_\alpha \left(\alpha e^{i\omega t/2} - \beta e^{-i\omega t/2}\right) + \frac{2i}{\omega}\left(\sigma^0\right)_{\dot{\alpha}}^{\alpha} Z_\alpha \tag{6}$$

where $\overset{\circ}{\phi}_\alpha$, Z_α and $\overline{Z}_{\dot{\beta}}$ are constant spinors and the frequency goes as: $\omega^2 \sim \frac{4}{|a|^2}$. The superfield solution for the fields (see the "square states" of Refs. [15–19]) that we are looking for, have the following form

$$g_{ab}(t) = e^{-\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1 t + c_2} e^{\xi \varrho(t)} g_{ab}(0) \tag{7}$$

with

$$\varrho(t) = \overset{\circ}{\phi}_\alpha \left[\left(\alpha e^{i\omega t/2} + \beta e^{-i\omega t/2}\right) - \left(\sigma^0\right)_{\dot{\alpha}}^{\alpha} \left(\alpha e^{i\omega t/2} - \beta e^{-i\omega t/2}\right)\right] + \frac{2i}{\omega}\left[\left(\sigma^0\right)_\alpha^{\dot{\beta}} \overline{Z}_{\dot{\beta}} + \left(\sigma^0\right)_{\dot{\alpha}}^{\alpha} Z_\alpha\right] \tag{8}$$

and

$$g_{ab}(0) = \langle \Psi(0)| L_{ab} |\Psi(0)\rangle \tag{9}$$

that is nothing more that the "square" of the state Ψ.[1] ($L_{ab} = \binom{a}{a^+}_{ab}$ with a and a^+ the standard creation and annihilation operators). The meaning of the expression (9) was given by the authors in Ref. [15] and can be resumed as:

(i) it is the "square" of the state Ψ and it is the *fundamental solution* of the square root of the interval (1), precisely describing a *trajectory* in the superspace [14–19];

(ii) for these states Ψ the *zero component of the current is not positively definite* given explicitly by

$$j_0(x) = 2E\Psi^\dagger \Psi$$

but for the states g_{ab}

$$j_0(x) = 2E^2 g^{ab} g_{ab}$$

then, $j_0(x)$ for the states g_{ab} is positively definite (the energy E appears squared);

(iii) from (ii), such states Ψ are related with *ordinary physical observables* only through they "square" g_{ab} in the sense of expressions as (9), and this fact is very important in order to explain the reason why these fractional spin states are not easy to see or to detect in the nature under ordinary conditions [15];

(iv) and fundamentally we will take under consideration here only the particular case of spin 2 because for this state the Hilbert space is dense and these states lead a *thermal* spectrum[2] [15, 24] (g_{ab} in the expression (9) has $s = 2$: each state Ψ contributes with a spin weight equal to one). Other interesting possibilities given by these type of coherent states solutions and they physical meaning, that can give some theoretical framework for more degrees of freedom for the graviton in the sense of [41–44], will be analyzed with details in a separate paper [24].

[1] This particular realization was initially introduced in Ref. [40] between the fundamental states $|\Psi\rangle$ in the initial time, where the subalgebra is the Heisenberg-Weyl algebra (with generators a, a^+ and $(n + \frac{1}{2})$).

[2] The other possibilities are squeezed states (non-thermal spectrum).

The g_{ab} at time t is given by the following expression [15, Appendix]

$$g_{ab}(t) = e^{-\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1' t + c_2'} e^{\xi\varrho(t)} |f(\xi)|^2 \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_{ab} \tag{10}$$

where α and α^* are the respective eigenvalues of the creation-annihilation operators a and a^+. And the dynamics for Ψ becomes now to

$$\Psi_\lambda(t) = e^{-\frac{1}{2}\left[\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1' t + c_2'\right]} e^{\frac{\xi\varrho(t)}{2}} |f(\xi)| \begin{pmatrix} \alpha^{1/2} \\ \alpha^{*1/2} \end{pmatrix}_\lambda \tag{11}$$

3 Relation with the (1 | 2) Superspace

We pass now to the description of the superspaces under consideration from the uniqueness of the possible solutions for the metric components, the supergroup structures defined by the possible group of motions and the possible physical interpretation of these results. The superspace $(1, 2)$ has one bosonic coordinate t and two majorana spinors: $x^\mu \equiv (t, \theta^1, \theta^2)$ (we use similar notation as in Refs. [20, 21]). The big group in which this superspace is contained is $OSP(3, 2)$, schematically as

$$\begin{pmatrix} O(3) & Q \\ Q & SP(2) \end{pmatrix}$$

The solution for the metric in this case is given by [20, 21]

$$\overline{g}_{ab} = g_{ab} e^{2\sigma(t,\theta)} \tag{12}$$

where the following superfield was introduced

$$\sigma(t, \theta) = A(t) + \theta^\beta B_\beta + \theta^\alpha \theta_\alpha F(t)$$

From the Einstein equations for the $(1 \mid 2)$ superspace we obtain the following set

$$\begin{cases} \dot{B}_\alpha + b_\alpha^\beta B_\beta - \dot{A} B_\alpha = 0 \\ \ddot{A} - \frac{1}{2}\dot{A}^2 + \frac{1}{2} B^\gamma B_\gamma = \frac{\lambda}{4}\left(e^{2A} - 1\right) \end{cases} \tag{13}$$

where $b_{\alpha\beta} = b_{\beta\alpha}$ is an arbitrary symmetric matrix. Making a suitable transformation in the first of above equations the explicit form of the B_γ field that we are looking for is

$$B^\gamma B_\gamma = \nu^\alpha \nu_\alpha e^{2A} \tag{14}$$

ν_α is a constant spinor and $\sqrt{b}$ was associated in the Ref. [21] with the mass. Inserting (14) in the second equation of the system (13) it leads the following new equation

$$\ddot{A}' - \frac{1}{2}\dot{A}'^2 = \frac{\lambda}{4}\left(e^{2A'} - 1\right) \tag{15}$$

where the transformation $A' = A - \frac{\nu^\alpha \nu_\alpha}{\lambda}$ was used. Notice that in the Ref. [21] the derivation of the solution of (15) was not explicitly explained, but however it is easy to see that can be reduced to the following expression

$$(\dot{W})^2 = \frac{\lambda}{4}\left(W^2 + \frac{1}{2W^2}\right) + C \tag{16}$$

with $W = e^{-\frac{A'}{2}}$ and C is an arbitrary constant. When $C = 0$ (16) is the equation of motion for a supersymmetric oscillator in the potential of the form $k(X^2 + \frac{1}{X^2})$, for which the group $O(3)$ is a dynamic symmetry group. Notice that from the point of view of a potential it is possible redefine it in order that C disappears, but the conservation of C is crucial for the determination of the families of solutions of the problem. This type of equations of motion for an oscillator with conformal symmetry was considered earlier in the non-supersymmetric case in [25]. The solutions for the possible values of the constant C are

$$C = 0 \to e^{-A} = \frac{\sqrt{2}}{2}\text{Sinh}\left(\sqrt{\lambda}t + \varphi_0\right), \quad \varphi_0 = \sqrt{\lambda}t_0,$$

$$\frac{8C^2}{\lambda^2} < 1 \to e^{-A} = \frac{\sqrt{2}}{2}\left[\text{Sinh}\left(\sqrt{\lambda}t + \varphi_0\right)\sqrt{1-\varkappa^2} - \varkappa\right], \quad \varkappa = \frac{2\sqrt{2}C}{\lambda},$$

$$\frac{8C^2}{\lambda^2} = 1 \to e^{-A} = \frac{\sqrt{2}}{2}\left[\frac{e^{(\sqrt{\lambda}t+\varphi_0)}}{\sqrt{2}} - \varkappa\right],$$

$$\sigma(t,\theta) = A(t) + \theta^\alpha B_\alpha \tag{17}$$

Notice that λ takes the place of the cosmological constant and is related with b by $b = \frac{-\lambda}{2}$. The superfield solution (17) is $N = 2$ (chiral or antichiral two components spinors), has conformal symmetry in the case $C = 0$ and is not unique: as was pointed out in the references [20, 21, 26] there exist a larger class of vacuum solutions. The dynamics of the solution is very simple as is easy to see from (17), that is not the case in the superspace $(1,3 \mid 1)$ as we showed in the previous section.

With the description of both superspaces above, we pass now to compare them in order to establish if a one to one mapping exists between these superspaces. By simple inspection we can see that the fermionic part of the superspace solutions (2) and (17) is mapped one to one, explicitly (for the $(1 \mid 2)$ superspace indexes 1 and 2 for α and β are understood).

$$\nu_\alpha = -2\beta\overset{\circ}{\phi}_\alpha,$$

$$2\sqrt{b} = \omega,$$

$$\theta^1 \leftrightarrow \overline{\theta}^{\dot{\alpha}}, \quad \theta^2 \leftrightarrow \theta^\alpha,$$

$$\left(\sigma^0\right)^\alpha_{\dot{\alpha}} \leftrightarrow b_1^2$$

if the following conditions over the four dimensional solution hold

$$\alpha = \beta, \quad Z_\alpha = \overline{Z}_{\dot{\beta}} = 0$$

For the bosonic part of the superfield solutions (17) and (4) no direct relation exists between them. Only taking the limit of the constants $|\mathbf{a}| \to \infty$ of the non-degenerate superspace $(1,3 \mid 1)$ (i.e. going to the standard $(1,3 \mid 1)$ superspace) the Gaussian solution (7) goes to the same type that the described in (17) for the (1 | 2) superspace, with $c_1 \approx \sqrt{\lambda}$ and $c_2 \approx \varphi_0$. And this fact is non trivial: because the chirality is explicitly restored in this limit as we can easily seen from (3) when $|\mathbf{a}| \to \infty, \omega^2 \to 0$. It is clear that the solution coming from four dimensional non-degenerate superspace is a physical one because represents a semiclassical (Gaussian) state of the Husimi's type [15, 27]. The important role played by the constants **a** and $\mathbf{a}^*$ in the extended line element (1) is localize the physical state in a precise region of the space-time, as is easily seen from expression (7). This fact can give some hints in order to explain and to treat from the mathematical point of view the mechanism of confinement, spontaneous compactification and other problems in high energy particle physics that can have a topological origin [24].

4 Supergravity as a Gauge Theory and the Origin of the Supermetric

The starting point is the super $SL(2C)$ superalgebra

$$[M_{AB}, M_{CD}] = \epsilon_C({}_A M_B)_D + \epsilon_D({}_A M_B)_C,$$

$$[M_{AB}, Q_C] = \epsilon_C({}_A Q_B), \quad \{Q_A, Q_B\} = 2M_{AB} \tag{18}$$

here the indices $A, B, C\ldots$ stay for $\alpha, \beta, \gamma\ldots(\dot{\alpha}, \dot{\beta}, \dot{\gamma}\ldots)$ spinorial indices: $(\alpha, \beta\dot{\alpha}, \dot{\beta}) = 1, 2(\dot{1}, \dot{2})$. We define the superconnection A due the following "gauging"

$$A^p T_p \equiv \omega^{\alpha\dot{\beta}} M_{\alpha\dot{\beta}} + \omega^{\alpha\beta} M_{\alpha\beta} + \omega^{\dot{\alpha}\dot{\beta}} M_{\dot{\alpha}\dot{\beta}} + \omega^{\alpha} Q_{\alpha} - \omega^{\dot{\alpha}} \overline{Q}_{\dot{\alpha}} \tag{19}$$

where (ωM) define a ten dimensional bosonic manifold[3] and $p \equiv$multi-index, as usual. Analogically the super-curvature is defined by $F \equiv F^p T_p$ with the following detailed structure

$$F(Mt)^{AB} = d\omega^{AB} + \omega^A_C \wedge \omega^{CB} + \omega^A \wedge \omega^B, \tag{20}$$

$$F(Q)^A = d\omega^A + \omega^A_C \wedge \omega^C \tag{21}$$

From () is not difficult to see, that there are a bosonic part and a fermionic part associated with the even and odd generators of the superalgebra. Our proposal for the action is

$$S = \int F^p \wedge \mu_p \tag{22}$$

where the tensor μ_p (that play the role of a super $SL(2C)$ diagonal metric) is defined as

$$\mu_{\alpha\dot{\beta}} = \zeta_\alpha \wedge \overline{\zeta}_{\dot{\beta}} \quad \mu_{\alpha\beta} = \zeta_\alpha \wedge \zeta_\beta \quad \mu_\alpha = \nu\zeta_\alpha \text{ etc.} \tag{23}$$

with $\zeta_\alpha(\overline{\zeta}_{\dot{\beta}})$ anticommuting spinors (suitable basis)[4] and ν the parameter of the breaking of super $SL(2C)$ to $SL(2C)$ symmetry of μ_p.

[3] Corresponding to the number of generators of $SO(4,1)$ or $SO(3,2)$ that define the group manifold.

[4] In general this tensor has the same structure that the Cartan-Killing metric of the group under consideration.

In order to obtain the dynamical equations of the theory, we proceed to perform the variation of the proposed action (22)

$$\begin{aligned}\delta S &= \int \delta F^p \wedge \mu_p + F^p \wedge \delta\mu_p \\ &= \int d_A\mu_p \wedge \delta A^p + F^p \wedge \delta\mu_p \end{aligned} \tag{24}$$

where d_A is the exterior derivative with respect to the super $SL(2C)$ connection and: $\delta F = d_A\delta A$ have been used. Then, as a result, the dynamics are described by

$$d_A\mu = 0, \quad F = 0 \tag{25}$$

The first equation said us that μ is covariantly constant with respect to the super $SL(2C)$ connection: this fact will be very important when the super $SL(2C)$ symmetry breaks down to $SL(2C)$ because $d_A\mu = d_A\mu_{AB} + d_A\mu_A = 0$, a soldering form will appear. The second equation give the condition for a super Cartan connection $A = \omega^{AB} + \omega^A$ to be flat, as is easily to see from the reductive components of above expressions

$$\begin{aligned} F(M)^{AB} &= R^{AB} + \omega^A \wedge \omega^B = 0 \\ F(Q)^A &= d\omega^A + \omega^A_C \wedge \omega^C = d_\omega \omega^A = 0 \end{aligned} \tag{26}$$

where now d_ω is the exterior derivative with respect to the $SL(2C)$ connection and $R^{AB} \equiv d\omega^{AB} + \omega^A_C \wedge \omega^{CB}$ is the $SL(2C)$ curvature. Then

$$F = 0 \Leftrightarrow R^{AB} + \omega^A \wedge \omega^B = 0 \quad \text{and} \quad d_\omega\omega^A = 0 \tag{27}$$

the second condition says that the $SL(2C)$ connection is super-torsion free. The first says not that the $SL(2C)$ connection is flat but that it is homogeneous with a cosmological constant related to the explicit structure of the Cartan forms ω^A, as we will see when the super $SL(2C)$ action is reduced to the Volkov-Pashnev model.

4.1 The Geometrical Reduction: Origin of the Extended Supermetric

The supermetric under consideration, proposed by Volkov and Pashnev in [14], can be obtained from the super $SL(2C)$ action via the following procedure:

(i) the Inönu-Wigner contraction [45] in order to pass from $SL(2C)$ to the super-Poincare algebra (corresponding to the original symmetry of the model of Refs. [14, 15]), then, the even part of the curvature is splitted into a $\mathbb{R}^{3,1}$ part $R^{\alpha\dot\beta}$ and a $SO(3,1)$ part $R^{\alpha\beta}(R^{\dot\alpha\dot\beta}t)$ associated with the remaining six generators of the original five dimensional $SL(2C)$ group. This fact is easily realized knowing that the underlying geometry is reductive: $SL(2C) \sim SO(4,1) \to SO(3,1) + \mathbb{R}^{3,1}$, and rewriting the superalgebra (18) as

$$\begin{array}{lll} [M,M] \sim M & [M,\Pi] \sim \Pi & [\Pi,\Pi] \sim M \\ [M,S] \sim S & [\Pi,S] \sim S & \{S,S\} \sim M + \Pi \end{array} \tag{28}$$

(with $\Pi \sim M_{\alpha\dot\beta}$ and $M \sim M_{\alpha\beta}(M_{\dot\alpha\dot\beta})$) and rescales $m^2\Pi = P$ and $mS = Q$, in the limit $m \to 0$ one recovers the super Poincare algebra. Notice that one does not rescale M since one want to keep $[M,M] \sim M$ Lorentz algebra (that also is symmetry of (1))

(ii) the spontaneous break down of the super $SL(2C)$ to the $SL(2C)$ symmetry of μ_p(e.g., $\nu \to 0$ in μ_p) of such manner that the even part of the super $SL(2C)$ action $F(M)^{AB}$ remains.

After these processes have been explicitly realized, the even part of the original super $SL(2C)$ action (now super-Poincare invariant) can be related with the original metric (1) as follows:

$$R(M) + R(P) + \omega^{\alpha}\omega_{\alpha} - \omega^{\dot{\alpha}}\omega_{\dot{\alpha}} \to \omega^{\mu}\omega_{\mu} + \mathbf{a}\omega^{\alpha}\omega_{\alpha} - \mathbf{a}^{*}\omega^{\dot{\alpha}}\omega_{\dot{\alpha}} \,|_{VP} \tag{29}$$

Notice that there is mapping $R(M)+R(P) \to \omega^{\mu}\omega_{\mu} \,|_{VP}$ that is well defined and can be realized of different forms, and the map of interest here $\omega^{\alpha}\omega_{\alpha} - \omega^{\dot{\alpha}}\omega_{\dot{\alpha}} \to \mathbf{a}\omega^{\alpha}\omega_{\alpha} - \mathbf{a}^{*}\omega^{\dot{\alpha}}\omega_{\dot{\alpha}} \,|_{VP}$ that associate the Cartan forms of the original super $SL(2C)$ action (22) with the Cartan forms of the Volkov-Pashnev supermodel: $\omega^{\alpha} = (\mathbf{a})^{1/2}\omega^{\alpha} \,|_{VP}$; $\omega^{\dot{\alpha}} = (\mathbf{a}^{*})^{1/2}\omega^{\dot{\alpha}} \,|_{VP}$. Then, the origin of the coefficients $\mathbf{a}$ and $\mathbf{a}^{*}$becomes clear from the geometrical point of view.

What about physics? From the first condition in (27) and the association (29) it is not difficult to see that, as in the case of the spacetime cosmological constant $\Lambda : R = \frac{\Lambda}{3} e \wedge e (e \equiv \mathit{space} - \mathit{time\ tetrad})$, there is a cosmological term from the superspace related to the complex parameters $\mathbf{a}$ and $\mathbf{a}^{*} : R = -(\mathbf{a}\omega^{\alpha}\omega_{\alpha} - \mathbf{a}^{*}\omega^{\dot{\alpha}}\omega_{\dot{\alpha}})$ and is easily to see from the minus sign in above expression, why for supersymmetric (supergravity) models is more natural to use $SO(3, 2)$ instead $SO(4, 1)$.

On the associated spinorial action in the action (22), notice that the role of this part is constrained by the nature of $\nu\zeta_{\alpha}$ in μ_p:

(i) If they are of the same nature of the ω^{α} this term is a total derivative, has not influence into the equations of motion, then the action proposed by Volkov and Pashnev in [14, 15] has the correct fermionic form.

(ii) If they are not with the same $SL(2C)$ invariance that the ω^{α}, the symmetry of the original model has been modified. In this direction a relativistic supersymmetric model for particles was proposed in Ref. [46] considering an N-extended Minkowski superspace and introducing central charges to the superalgebra. Hence the underlying rigid symmetry gets enlarged to N-extended super-Poincare algebra. Considering for our case similar superextension that in Ref. [46] we can introduce the following new action

$$\begin{aligned} S &= -m \int_{\tau 1}^{\tau 2} d\tau \sqrt{\omega_{\mu}^{\circ}\overset{\circ}{\omega}{}^{\mu} + a\dot{\theta}^{\alpha}\dot{\theta}_{\alpha} - a^{*}\dot{\overline{\theta}}{}^{\dot{\alpha}}\dot{\overline{\theta}}_{\dot{\alpha}} + i(\theta^{\alpha i}A_{ij}\dot{\theta}_{\alpha}^{j} - \overline{\theta}^{\dot{\alpha} i}A_{ij}\dot{\overline{\theta}}{}_{\dot{\alpha}}^{j})} \\ &= \int_{\tau 1}^{\tau 2} d\tau L\left(x, \theta, \overline{\theta}\right) \end{aligned} \tag{30}$$

that is the super-extended version of the superparticle model proposed in [14], with the first order fermionic part. The matrix tensor A_{ij} introduce the symplectic structure of such manner that now $\zeta_{\alpha i} \sim A_{ij}\theta_{\alpha}^{j}$ is not covariantly constant under d_{ω}. Notice that the "Dirac-like" fermionic part is obviously *inside* the square root because it is part of the full curvature, fact that was not advertised by the authors in [46] that not take account on the geometrical origin of the action. The interesting point is perform the same quantization that in the first part of this research [15] in order to obtain and compare the spectrum of physical states with the obtained in Ref. [46]. This issue will be presented elsewhere [24].

5 "Warped" Gravity Models, Confinement and the Supermetric

It is well known that large extra dimensions offer an opportunity for a new solution to the hierarchy problem [28, 29]. Field theoretical localization mechanisms for scalar and fermions [30] as well as for gauge bosons [31, 32] were found. The crucial ingredient of this scenario is a brane on which standard model particles are localized. In string theory, fields can naturally be localized on D-branes due to the open strings ending on them [33]. Up until recently, extra dimensions had to be compactified, since the localization mechanism for gravity was not known. It was suggested in Refs. [34, 35] that gravitational interactions between particles on a brane in uncompactified five dimensional space could have the correct four dimensional Newtonian behaviour, provided that the bulk cosmological constant and the brane tension are related. Recently, it was found by Randall and Sundrum that gravitons can be localized on a brane which separates two patches of AdS_5 space-time [36, 37]. The necessary requirement for the four-dimensional brane Universe to be static is that the tension of the brane is fine-tuned to the bulk cosmological constant [34–37]. By the other hand, recent papers present an interesting model in which the extra dimensions are used only as a mathematical tool taking advantage of the AdS/CFT correspondence that claims that the 5D warped dimension is related with a strongly coupled 4D theory [38].

A remarkable property of the solution given by the expression (7) is that the physical state $g_{ab}(x)$ is localized in a particular position of the space-time. The supermetric coefficients $\mathbf{a}$ and $\mathbf{a}^*$ play the important role of localize the fields in the bosonic part of the superspace in similar and suggestive form as the well known "warp factors" in multidimensional gravity [22] for a positive (or negative) tension brane. But the essential difference is, because the $\mathbb{C}$-constants $\mathbf{a}$ and $\mathbf{a}^*$ coming from the $B_{L,0}$ (even) fermionic part of the superspace under consideration, not additional and/or topological structures that break the symmetries of the model (i.e. reflection Z_2-symmetry) are required: the natural structure of the superspace produces this effect.

Also it is interesting to remark here that the Gaussian type solution (7) is very well defined physical state in a Hilbert space [15, 27] from the mathematical point of view (analyticity and continuity), contrarily to the case $u(y) = ce^{-H|y|}$ given in [22] that, although were possible to find a manner to include it in any Hilbert space, is strongly needed to take special mathematical and physical particular assumptions whose meaning is obscure. The important point to remark here is that when we describe from the mostly geometrical grounds any physical system through SU(1,1) Coherent or Squeezed states (CS or SS, as in the case given by expression (7)), the orbits will appear as the intersections of curves that represent constant-energy surfaces, with one sheet of a two sheeted hyperboloid- the curved phase space of SU(1,1) or Lobachevsky plane- in the space of averaged algebra generators. In the specific case treated in this paper, the group containing the SU(1,1) as subgroup linear and bilinear functions of the algebra generators can factorize operators as the Hamiltonian or the Casimir operator (when averaged with respect to group CS or SS), defining corresponding curves in the averaged algebra space. If we notice that the validity of the Ehrenfest's theorem for CS (SS) implies that, if the exact dynamics is confined to the SU(1,1) hyperboloid, it necessarily coincides with the variational motion, the variational motion that comes from the Euler-Lagrange equations for the lagrangian [27]

$$\mathcal{L} = \left\langle z \middle| i\frac{\widehat{\partial}}{\partial t} - \widehat{H} \middle| z \right\rangle$$

with $|z\rangle = |\Psi\rangle$, as is evident to see. It is interesting to note also that similar picture holds in the context of the pseudospin SU(1,1) dynamics in the frame of the mean field approximation induced by the variational principle on nonlinear Hamiltonians.

Table 1

Spacetime	(5-D) gravity + Λ	Superspace $(1,3 \mid 1)$
Interval	$ds^2 = A(y)dx^2_{3+1} - dy$	$ds^2 = \omega^\mu \omega_\mu + \mathbf{a}\omega^\alpha \omega_\alpha - \mathbf{a}^* \omega^{\dot\alpha} \omega_{\dot\alpha}$
Equation	$[-\partial_y^2 - m^2 e^{H\|y\|} + H^2 - 2H\delta(y)]$ $\times u(y) = 0$	$[\|a\|^2(\partial_0^2 - \partial_i^2) + \frac{1}{4}(\partial_\eta - \partial_\xi + i\partial_\mu(\sigma^\mu)\xi)^2$ $-\frac{1}{4}(\partial_\eta + \partial_\xi + i\partial_\mu(\sigma^\mu)\xi)^2 + m^2]^{ab}_{cd} g_{ab} = 0$
Solution	$u(y) = ce^{-H\|yt\|}, \quad H \equiv \sqrt{-\frac{2\Lambda}{3}} = \frac{\|T\|}{M^3}$	$g_{ab}(x) = e^{-(\frac{m}{\|\mathbf{a}\|})^2 x^2 + c_1' x + c_2'} e^{\xi \varrho(x)} \|f(\xi)\|^2 \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_{ab}$

The comparison with the case of 5-dimensional gravity plus cosmological constant [22] is given in Table 1.

Here, in order to make our comparison consistent, the proposed superspace has $d = 4$ bosonic coordinates and the extended superspace solution for can depend, in principle, on any or all the 4-dimensional coordinates: $x \equiv (t, \overline{x})$, $c_1' x \equiv c'_{1\mu} x^\mu$ and c_2' scalar (e.g., the t coordinate in expression (7)); for $n \neq 0$ it depends on the n-additional coordinates.

Notice the following important observations:

(i) that for that the solution in the 5-dimensional gravity plus Λ case, the explicit presence of the cosmological term is necessary for the consistency of the model: the "fine-tuning" $H \equiv \sqrt{-\frac{2\Lambda}{3}} = \frac{|T|}{M^3}$, where T is the tension of the brane and M^3 is the constant of the Einstein-Hilbert $+\Lambda$ action.

(ii) about the localization of the fields given by the particular superspace treated here, the Z_2 symmetry is non-compatible with the solution that clearly is not chiral or antichiral. This fact is consistent with the analysis given for a similar superspace that the considered here in Refs. [15, 39] where the solutions are superprojected in a sector of the physical states that is not chiral or antichiral.

(iii) because for $d = 4$ our solution (7) is attached on the $3 + 1$ space-time but the localization occurs on the time coordinate (on in any of the remanent 3 space coordinates) the physics seems to be very different with respect to the warped gravity model where the field equation in final form for the 5-dimensional gravity depends on the extra dimension.[5] This $n = 0$ case can give some hints for the theoretical treatment of the confinement mechanism with natural breaking of the chiral symmetry in high energy physics (e.g., instant on liquid models, etc.).

(iv) for $d = 5$, our model with the solution depending on the extra coordinate, the situation changes favorably: the localization of the field is in the additional bosonic coordinate (as the graviton in the RS type model) but with all the good properties of the solution (7) already mentioned in the beginning of this paragraph. The multidimensional case will be explicitly studied in [24].

From the points discussed above and the "state of the art" of the problem, we seen the importance of to propose new mechanisms and alternative models that can help us to understand and to handle the problem. Then, it is not difficult to think to promove the particular supermetric under study towards to build a strongly coupled 4D model, using this particular $N = 1$ toy superspace. We will treat this issue with great detail in a further work [24].

[5] E.g., in the Randall-Sundrum model the graviton is localized in the extra dimension.

6 Concluding Remarks

In the present paper we have analyzed from the point of view of the symmetries and the obtained vacuum solutions the superspace $N = 1$ non-degenerate metric proposed by Volkov and Pashnev in [14]. This particular model, although its high simplicity, present a much richer structure than the others degenerate standard superspaces because it contains the complex parameters $\mathbf{a}$ and $\mathbf{a}^*$ that make it different. The role played by the complex parameters $\mathbf{a}$ and $\mathbf{a}^*$ can be resumed as follows:

(i) the $\mathbb{C}$-parameters $\mathbf{a}$ and $\mathbf{a}^*$ fix the field in a specific sector of the even part ($B_{L,0}$) of the supermanifold;
(ii) these parameters, that are responsible of the non trivial part of the model, break the chiral symmetry of the field solution. The chiral symmetry is restored when the metric in question degenerate in the limit $|\mathbf{a}| \to \infty$ (with all other parameters of the model fixed);
(iii) the fields remain attached in a specific region of the spacetime when the supersymmetry of the model is completely broken even if all the fermions are switched off.
(iv) we have analyzed and compared from the point of view of the obtained solutions the superspace $(1 \mid 2)$ with the particular superspace $(1, 3 \mid 1)$ proposed by Volkov and Pashnev [14, 16], compactified to one dimension and restricted to the pure time-dependent case. The possibility that the non-degenerate superspace $(1, 3 \mid 1)$ with extended line element is reduced to the superspace $(1 \mid 2)$ is subject to the condition $|\mathbf{a}| \to \infty$. The fermionic part of both superspaces is mapped one to one by mean of a suitable definition of the fermionic variables and coefficients.

From the geometrical and group theoretical point of view the results are the following:

(v) the supermetric can be derived from a gauge theory of supergravity based in the super $SL(2C)$ group
(vi) the complex parameters $\mathbf{a}$ and $\mathbf{a}^*$ play similar role that the cosmological constant Λ in the ordinary spacetime models. Then, add a Λ constant by hand is not necessary in this supersymmetric model
(vii) a new generalization of the Volkov-Pashnev superparticle is presented for $N > 1$ supersymmetry where the first order fermionic term appear explicitly in the action.

In comparison with the 5-dimensional gravity plus cosmological constant of Ref. [22], the simple supersymmetric model under analysis here (now with n-extra bosonic coordinates) has the following advantages:

(viii) for $d = 4$ the model, although is a very good candidate for a confinement mechanism with natural breaking of the chiral symmetry in high energy physics (e.g., instant on liquid models, etc.), cannot be compared directly with the Randall-Sundrum model because the localization of the fields are not in the bosonic extra-dimension (the physic are different in both cases).
(ix) for $d \geq 4$ the mechanism of localization of the fields in the bosonic 4-dimensional part of the supermanifold does not depends on the cosmological constant (the detailed description of the multidimensional case will appear elsewhere [24]);
(x) the fields attached are Gaussian type solutions (7) very well defined physical states in a Hilbert space from the mathematical (analicity and continuity) point of view, with all the physical properties of the mean values of operators between Coherent states [27] (e.g., good agreement with the classical limit and temporal stability), contrarily to the case $u(y) = ce^{-H|y|}$ given in [22];

(xi) not additional and/or topological structures that break the symmetries of the model (i.e. reflection Z_2-symmetry) are required to attach the fields: the natural structure of the superspace produces this effect through the $\mathbb{C}$-parameters **a** and **a***.

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Appendix

The dynamics of the $|\Psi\rangle$ fields, in the representation that we are interested in, can be simplified considering these fields as coherent states in the sense that are eigenstates of a^2 [27]

$$\left|\Psi_{1/4}(0,\xi,q)\right\rangle = \sum_{k=0}^{+\infty} f_{2k}(0,\xi)|2k\rangle = \sum_{k=0}^{+\infty} f_{2k}(0,\xi)\frac{(a^\dagger)^{2k}}{\sqrt{(2k)!}}|0\rangle$$

$$\left|\Psi_{3/4}(0,\xi,q)\right\rangle = \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)|2k+1\rangle = \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)\frac{(a^\dagger)^{2k+1}}{\sqrt{(2k+1)!}}|0\rangle \tag{31}$$

From a technical point of view these states are a one mode squeezed states constructed by the action of the generators of the SU(1,1) group over the vacuum. For simplicity, we will take all normalization and fermionic dependence or possible CS fermionic realization, into the functions $f(\xi)$. Explicitly at $t=0$

$$\begin{aligned}\left|\Psi_{1/4}(0,\xi,q)\right\rangle &= f(\xi)\,|\alpha_+\rangle \\ \left|\Psi_{3/4}(0,\xi,q)\right\rangle &= f(\xi)\,|\alpha_-\rangle\end{aligned} \tag{32}$$

where $|\alpha_\pm\rangle$ are the CS basic states in the subspaces $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$ of the full Hilbert space. In the case of the physical state that we are interested in, we used the HW realization for the states Ψ

$$|\Psi\rangle = \frac{(\xi)}{2}(|\alpha_+\rangle + |\alpha_-\rangle) = f(\xi)|\alpha\rangle \tag{33}$$

where, however, the linear combination of the states $|\alpha_+\rangle$ and $|\alpha_-\rangle$ span now the full Hilbert space (dense) being the correspond λ to the CS basis. The "square" state at $t=0$ are

$$g_{ab}(0) = \langle\Psi(0)|L_{ab}|\Psi(0)\rangle = \langle\Psi(0)|\begin{pmatrix} a \\ a^+ \end{pmatrix}_{ab}|\Psi(0)\rangle \tag{34}$$

$$= f^*(\xi)f(\xi)\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_{ab} \tag{35}$$

The algebra (topological information of the group manifold) is "mapped" over the spinors solutions through the eigenvalues α and α^*. Notice that the constants c_1' c_2' in the exponential functions in expressions (10) and (11) can be easily determined as functions of the frequency ω as in Ref. [27] for the Schrödinger equation.

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